introduce the Oz axis so that $\zeta(\varphi) = 0$ for $\varphi = \infty$, then $c = 0$ in (2.1). We assume that $|\zeta'(\varphi)|$ is small; then the linearized equation (2.1) has the following solution for $g \neq 0$:

$$\zeta_0 = c_0 \sqrt{\sigma + T\varphi} K_1\left(2 \sqrt{\frac{\rho g}{T^2}(\sigma + T\varphi)}\right)$$

[$K_1(t)$ is the modified Bessel function]. The constant c_0 is determined easily by the value of the wetting angle at the point of fluid contact with the wall.

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NONLINEAR CRITICAL LAYER AND FORMATION OF LINEAR VORTICES WITH REACTION OF WAVES IN SHEAR FLOWS

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Recently in hydromechanics there has been a considerable increase in interest in the problem of transition from laminar to turbulent flow [1-3]. Of considerable importance for explaining processes occurring during transition to turbulence in shear flows is analysis of the nonlinear structures occurring as a result of the development of hydrodynamic instability. Experiments show that in boundary flows such as a boundary layer and Poiseuille flow, occurrence of turbulence is connected with formation of Λ-vortices characterized by a considerable linear (in relation to flow direction) component of vorticity [4-9]. In [10] attention was drawn to the related connection of these vortices with large-scale bounded structures observed in the region near the wall of developed turbulent flow.

The theory of Benney and Lin [11, 12] connects oscillation of linear vorticity in transitional flow with an increase in it of pairs of inclined (three-dimensional) waves having the same phase velocity and linear components of wave vectors. The instantaneous profile of the transverse velocity determined in [12] within the framework of linear approximation demonstrates two reversals of velocity for the period of the wave, whereas in experiments [5, 10] sequences of profiles are observed with one reversal which corresponds to passage through a stationary observation point for one vortex formation in the period of the wave. In this work a study is made of essentially nonlinear vortex structures occurring in a critical layer (CL) of laminar flow with resonance reaction of two-dimensional and inclined waves increasing in it. Analysis is built up within the framework of an asymptotic approach rest-

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ing on use of the smallness of CL thickness. In this way nonlinearity may only be considerable within the limits for the thin CL. At the present time main attention is being devoted to studying two-dimensional CL (see, e.g., [13-15]). The problem of the CL structure in the case of a single inclined wave is also reduced to a two-dimensional problem [13]. Suggested below is an explanation of the mechanism for forming Λ-vortices based on analyzing the dynamics of a three-dimensional nonlinear CL in ideal flow.

The mechanism for occurrence of a wave triplet, introduced heuristically by Benney and Lin, is explained in [16]. According to [16], generation of inclined waves is due to their resonance reaction with the second harmonic of a two-dimensional wave which is in synchronism with this wave. The diagram combining wave vectors with this process is shown in Fig. 1a ($\mathbf{k}_0$ and $\mathbf{k}_{1,2}$ are wave vectors for two-dimensional and inclined waves, respectively). In the case of identical slopes between vectors $\mathbf{k}_{1,2}$ and $\mathbf{k}_0$ (symmetrical triplet) from the synchronism condition it follows that there is equality of frequencies and wave phase velocities $\mathbf{k}_{0,1,2}$ (it is achieved by selecting the angle between wave vectors for two-dimensional and inclined waves). A situation often occurs in an experiment when the natural wave for flow with wave vector $2\mathbf{k}_0$ attenuates strongly. With this condition synchronism for generation of the second harmonic is disturbed and it should be considered as an induced wave. A four-wave process arises which is described in third-order theory for perturbation of wave amplitudes. Since the amplitude of the second harmonic in this case is small, pulsations of velocity and pressure in the flow are determined approximately by the wave triplet $\mathbf{k}_2$, all waves of which have identical frequencies. The possibility of generating pairs of inclined waves with their direct resonance reaction with a two-dimensional wave was considered theoretically in [17].

With symmetrical location of wave vectors $\mathbf{k}_{1,2}$ relative to $\mathbf{k}_0$ the frequencies of inclined waves equal half the frequency of the two-dimensional wave. Combination of wave vectors with this "subharmonic" reaction is shown in Fig. 1b. Experimental proof of realization for subharmonic reaction in the boundary layer is given in [18]. In [19] "triharmonic" self-sustained regimes were obtained corresponding to saturated explosive instability of wave triplets in a plane channel. The processes mentioned are basic with classical and subharmonic regimes for failure of laminar flow in the boundary layer [9, 18]. A classical (single-frequency) regime normally arises with large values of two-dimensional wave amplitude, which may be explained by the lower effectiveness of four-wave reaction compared with three-wave reaction with low wave amplitudes.

With realization in a flow of single-frequency and subharmonic reaction of waves, all of them have the same phase velocity, and in the flow velocity profile U(y) one (common for all waves) resonance point $y = y_c$ $[U(y_c) = c]$ arises. Considered below is evolution of a CL occurring in ideal flow in the vicinity of section $y = y_c$. Approximation of ideal flow makes it possible to draw important qualitative conclusions relating to processes with presence of low viscosity, and in addition it is of interest in view of use with numerical modeling of flows [20]. Amplitudes for all waves are assumed to be constant. Consideration of the slowly varying dependence of amplitude on time is not reflected in the form of equations for the CL itself (see similarly in [14]), also necessary in describing reverse reaction of CL in waves, which in this case is not considered. With finite viscosity and a change in wave amplitude in time it is possible to introduce hierarchy for CL thickness [14]. The case of a nonlinear CL in ideal flow corresponds to predominance of nonlinear CL thickness. It is noted that approximation of a thin CL is correct with weak instability typical mainly for flow near walls. In flows with strong instability (e.g., in a shear layer) in the linear stage of development of instability CL thickness, calculated by the equation for a thin CL [14], appears comparable with the scale of shear velocity in primary flow.

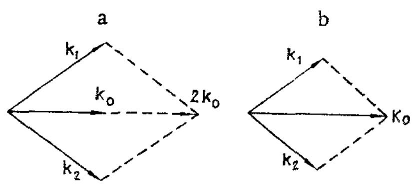

Fig. 1

1. We use a dimensionless form of writing hydrodynamic equations for an ideal incompressible liquid (normalizing introduced in a normal form for stability theory). Axes x and z of a Cartesian coordinate system are directed along and across the flow, and axis y is directed along the normal to the flow plane. By representing the linear component of the velocity vector v in the form $v_1 = U(y) + \tilde{v}_1$, we obtain

$$\frac{\partial \tilde{v}_1}{\partial t} + U\frac{\partial \tilde{v}_1}{\partial x} + v_2\frac{dU}{dy} + \frac{\partial p}{\partial x} = -\tilde{v}_1\frac{\partial \tilde{v}_1}{\partial x} - v_2\frac{\partial \tilde{v}_1}{\partial y} - v_3\frac{\partial \tilde{v}_1}{\partial z} \equiv F_1, \tag{1.1}$$
$$\frac{\partial v_2}{\partial t} + U\frac{\partial v_2}{\partial x} + \frac{\partial p}{\partial y} = -\tilde{v}_1\frac{\partial v_2}{\partial x} - v_2\frac{\partial v_2}{\partial y} - v_3\frac{\partial v_2}{\partial z} \equiv F_2,$$
$$\frac{\partial v_3}{\partial t} + U\frac{\partial v_3}{\partial x} + \frac{\partial p}{\partial z} = -\tilde{v}_1\frac{\partial v_3}{\partial x} - v_2\frac{\partial v_3}{\partial y} - v_3\frac{\partial v_3}{\partial z} \equiv F_3,$$
$$\frac{\partial \tilde{v}_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

(p is dimensionless pressure). We introduce small parameter ε characterizing the serial value of oscillation amplitude, and we shall build up solution (1.1) in the form of an expansion in powers of ε:

$$\tilde{v}_1 = \varepsilon\tilde{v}_1^{(1)} + \varepsilon^2\tilde{v}_1^{(2)} + \ldots,\ v_j = \varepsilon v_j^{(1)} + \varepsilon^2 v_j^{(2)} + \ldots\ (j = 2, 3),$$
$$p = \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \ldots \tag{1.2}$$

For nonlinear terms in (1.1) we obtain, respectively, $F_j = \varepsilon F_j^{(1)} + \varepsilon^2 F_j^{(2)} + \ldots$. In view of Squire's transform the search for y-components of velocity in an inclined wave is reduced to solving the similar problem for a two-dimensional wave with wave number $\alpha_0 = \sqrt{\alpha^2 + \beta^2}$ (α and β are wave vector components along axes x and z). By prescribing to a first approximation one of the wave triplets shown in Fig. 1, we have

$$v_2^{(1)} = \sum_{\alpha,\beta}(A_\pm\varphi_a + B_\pm\varphi_b)\,e^{i\theta} + \text{c.c.}, \tag{1.3}$$

where $\theta = \alpha(x - ct) + \beta z$; φ_a and φ_b are Tollmien functions [13] forming a fundamental set of characteristic functions of the Rayleigh equation for a two-dimensional wave with wave number α_0 and phase velocity c; $A_\pm(\alpha, \beta)$ and $B_\pm(\alpha, \beta)$ are constants; summing is carried out for values of α and β relating to one of the wave triplets. Indices + and − relate to regions $y > y_c$ and $y < y_c$, found on different sides of the resonance point at which derivatives with respect to y undergo a break. We shall designate by symbol ∧ amplitudes of Fourier harmonics for variables; $\hat{v}_j = \langle v_j \exp(-i\theta)\rangle$, etc. (⟨...⟩ are average for space-time oscillations). By substituting (1.3) in (1.2), equating coefficients with the same powers of ε, and expressing amplitudes $\hat{p}^{(\ell)}$, $\hat{\tilde{v}}_1^{(\ell)}$, and $\hat{v}_3^{(\ell)}$ in terms of $\tilde{v}_2^{(\ell)}$, we find that

$$\frac{d\hat{v}_2}{dy} - \left(\alpha_0^2 + \frac{U''}{U-c}\right)\hat{v}_2 = \frac{1}{U-c}\left(\frac{i\alpha_0^2}{\alpha}\hat{F}_2 - \frac{d\hat{F}_1}{dy} - \frac{\beta}{\alpha}\frac{d\hat{F}_3}{dy}\right), \tag{1.4}$$
$$\hat{p} = \frac{\alpha}{i\alpha_0^2}\left[(U-c)\frac{d\hat{v}_2}{dy} - U'\hat{v}_2\right] + \frac{\alpha}{i\alpha_0^2}\left(\hat{F}_1 + \frac{\beta}{\alpha}\hat{F}_3\right),$$
$$\hat{\tilde{v}}_1 = -\frac{1}{i\alpha_0^2\alpha}\left(\alpha^2\frac{d\hat{v}_2}{dy} + \frac{\beta^2 U'}{U-c}\hat{v}_2\right) - \frac{\beta}{i\alpha_0^2(U-c)}\left(\hat{F}_3 - \frac{\beta}{\alpha}\hat{F}_1\right),$$
$$\hat{v}_3 = \frac{\beta}{i\alpha_0^2}\left(-\frac{d\hat{v}_2}{dy} + \frac{U'}{U-c}\hat{v}_2\right) + \frac{\alpha}{i\alpha_0^2(U-c)}\left(\hat{F}_3 - \frac{\beta}{\alpha}\hat{F}_1\right),$$

where primes indicate derivatives of U with respect to y and for brevity the upper index (ℓ), ℓ = 1, 2, 3, ... is omitted for all amplitudes. The sequence of transformations in changing from (1.1) to (1.4) is easily reconstructed from the form of combinations of $\hat{F}_j$ in the right-hand parts of (1.4). Since $F_j^{(1)} \equiv 0$, with ℓ = 1 from (1.4) the results of linear theory follow. By using relationship (1.4) for harmonics with α ≠ 0 and expressions emerging directly from (1.1) for components with α = 0, it is possible to determine the terms of expansion (1.2) in explicit form.

It can be seen from (1.3) and (1.4) that close to the resonance point ($y - y_c \to 0$) it is possible to consider nonlinearity within the scope of the normal method of disturbances with respect to ε. A nonlinear CL with thickness $\sim\varepsilon^{1/2}$ arises in the flow. By determining asymptotically the first two terms in expansion (1.2) with $y - y_c \to 0$ and changing over in them to a variable CL, $Y = (y - y_c)/\varepsilon^{1/2}$, we obtain an expansion for harmonic amplitudes with $Y \to \pm\infty$ (external expansions for CL):

$$\widehat{p} = \varepsilon B_{p\pm} + \varepsilon^2 \ln \varepsilon^{1/2} C_\pm + O(\varepsilon^2), \tag{1.5}$$
$$\widehat{v}_2 = \varepsilon\left[-\frac{i\alpha_0^2}{\alpha U_c'} B_{p\pm} + O(1/Y^2)\right] + O(\varepsilon^{3/2}\ln\varepsilon^{1/2}),$$
$$\widehat{\widetilde{v}}_1 = \varepsilon^{1/2}\left[\frac{\beta^2}{\alpha^2 U_c' Y} B_{p\pm} + O(1/Y^3)\right] + O(\varepsilon\ln\varepsilon^{1/2}),$$
$$\widehat{v}_3 = \varepsilon^{1/2}\left[-\frac{\beta}{\alpha U_c' Y} B_{p\pm} + O(1/Y^3)\right] + O(\varepsilon\ln\varepsilon^{1/2}).$$

Here $B_{p\pm} = (i\alpha U_c'/\alpha_0^2)B_\pm$ is complex amplitude for pressure in the wave (α, β) with $y \to y_c \pm 0$ in units of ε; $C_\pm$ are constants which are expressed in terms of the value $B_{p\pm}$ with different α and β. Estimation of the rest of the terms in the main part of expansion (1.5) for velocity components are given for the case of a subharmonic triplet.

In order to describe a nonlinear CL we change over into (1.1) to variables Y, $\xi = x - ct$, and "slowly varying" time $\tau = \varepsilon^{1/2}t$ (ξ is coordinate downward for flow in the reckoning system connected with waves; normalizing of time is introduced from the condition of equality in the sequence of values for nonlinear and nonsteady terms in equations for the CL). By using an expansion for primary flow velocity in the reckoning system for waves $U - c = \varepsilon^{1/2}U_c'Y + {}^1/_2\varepsilon U_c''Y^2 + \ldots$ and taking notice of (1.5), we build up a solution for the equations obtained

$$p = \varepsilon P^{(0)} + \varepsilon^2 \ln \varepsilon^{1/2} P^{(1)} + \varepsilon^2 P^{(2)} + \ldots, \tag{1.6}$$
$$v_2 = \varepsilon V_2^{(0)} + \varepsilon^{3/2}\ln\varepsilon^{1/2} V_2^{(1)} + \varepsilon^{3/2} V_2^{(2)} + \ldots,$$
$$\widetilde{v}_1 = \varepsilon^{1/2}\widetilde{V}_1^{(0)} + \varepsilon\ln\varepsilon^{1/2}\widetilde{V}_1^{(1)} + \varepsilon\widetilde{V}_1^{(2)} + \ldots,$$
$$v_3 = \varepsilon^{1/2} V_3^{(0)} + \varepsilon\ln\varepsilon^{1/2} V_3^{(1)} + \varepsilon V_3^{(2)} + \ldots$$

It is easy to be certain that variable $P^{(0)}$ satisfies the equation $\partial P^{(0)}/\partial Y = 0$, and consequently it does not depend on Y. From the condition for combining (1.6) with external expansions (for harmonic amplitudes) we obtain $B_{p+}(\alpha, \beta) = B_{p-}(\alpha, \beta) \equiv B_p$. This rule automatically provides combination for variable $P^{(1)}$, which also satisfies the equation $\partial P^{(1)}/\partial Y = 0$. The expression for $P^{(0)}$ has the form

$$P^{(0)} = \sum_{\alpha,\beta} B_p(\alpha, \beta)\, e^{i\alpha\xi + i\beta z} + \text{c.c.} \tag{1.7}$$

Thus, the pressure field in a thin CL is determined by wave disturbances propagating in the main part of the flow. The set of equations connecting the main terms of expansion (1.6) takes the form

$$\frac{\partial\widetilde{V}_1^{(0)}}{\partial\tau} + \left(U_c'Y + \widetilde{V}_1^{(0)}\right)\frac{\partial\widetilde{V}_1^{(0)}}{\partial\xi} + V_2^{(0)}\left(U_c' + \frac{\partial\widetilde{V}_1^{(0)}}{\partial Y}\right) + V_3^{(0)}\frac{\partial V_1^{(0)}}{\partial z} = -\frac{\partial P^{(0)}}{\partial\xi}, \tag{1.8}$$
$$\frac{\partial V_3^{(0)}}{\partial\tau} + \left(U_c'Y + \widetilde{V}_1^{(0)}\right)\frac{\partial V_3^{(0)}}{\partial\xi} + \widetilde{V}_2^{(0)}\frac{\partial V_3^{(0)}}{\partial Y} + V_3^{(0)}\frac{\partial V_3^{(0)}}{\partial z} = -\frac{\partial P^{(0)}}{\partial z},$$
$$\frac{\partial\widetilde{V}_1^{(0)}}{\partial\xi} + \frac{\partial V_2^{(0)}}{\partial Y} + \frac{\partial V_3^{(0)}}{\partial z} = 0.$$

Following the general scheme for the method of combined asymptotic expansions, it is necessary to find solution of (1.8) which combines with the main part of expansion (1.5) with $Y \to \pm\infty$. It is possible to see that this solution is simultaneously a solution for the boundary problem with boundary conditions $\tilde{V}_1^{(0)} \to 0$, $V_3^{(0)} \to 0$, $V_2^{(0)} \to V_c^{(0)}$, where $V_c^{(0)}$ is oscillations of the y component of velocity $V_{(2)}^0$ in a flow with velocity $U_c'Y$ in an approximation of linear theory ($V_c^{(0)}$ coincides with limit $v_2^{(1)}$ with $y \to y_c$). In order to plot asymptotics for solution of this boundary problem with large $|Y|$ it is assumed that $Y = \eta/\mu$ and we use disturbance theory for small parameter $\mu \ll 1$. With an increase from variable η to Y series of disturbance theory with respect to parameter μ give an expansion with respect to 1/Y contained in the main part of (1.5). Statement of the boundary problem for (1.8) makes it possible to talk about resonance excitation and development of vortex disturbances in a uniformly swirling incompressible gas of particles not reacting with each other.

After introducing total linear flow velocity in the CL $V_1^{(0)} = U_c'Y + \tilde{V}_1^{(0)}$ and changing over to Lagrangian variables, the first two equations of set (1.8) change over to a closed set of equations for the "horizontal" (with components in the flow plane) component of individual particle movement:

$$\frac{dV_1^{(0)}}{d\tau} = -\frac{\partial P^{(0)}}{\partial \xi},\ \frac{d\xi}{d\tau} = V_1^{(0)},\ \frac{dV_3^{(0)}}{d\tau} = -\frac{\partial P^{(0)}}{\partial z},\ \frac{dz}{d\tau} = V_3^{(0)}. \qquad (1.9)$$

In view of the absence of a dependence for pressure field on Y and absence of self-conforming pressure, horizontal movement of liquid particles is autonomous and similar to movement of a set of material points in the potential $P^{(0)}(\xi, z)$. The normal velocity component in the CL is determined from the continuity equation in set (1.8).

By using (1.6) it is possible to write expansions for vorticity $\omega = \mathrm{curl}\,\mathbf{v}$: $\omega_1 = \omega_1^{(0)} + \varepsilon^{1/2}\ln\varepsilon^{1/2}\omega_1^{(1)} + \ldots$, $\omega_3 = \omega_3^{(0)} + \varepsilon^{1/2}\ln\varepsilon^{1/2}\omega_3^{(1)} + \ldots$, $\omega_2 = \varepsilon^{1/2}\omega_2^{(0)} + \varepsilon\ln\varepsilon^{1/2}\omega_2^{(1)} + \ldots$. In this way $\omega_1^{(0)}\ \partial V_3^{(0)}/\partial Y$, $\omega_3^{(0)} = -\partial V_1^{(0)}/\partial Y$, $\omega_2^{(0)} = \partial V_1^{(0)}/\partial z - \partial V_3^{(0)}/\partial \xi$, i.e., the connection between tangents to the flow plane with velocity and vorticity components appears to be the same as for a unidimensional vortex layer. From the conservation rule vortices should link vorticity distribution with the relative change in material element length $\delta\mathbf{l}$, lying on the vortex line [21]:

$$\frac{\omega(\tau)}{|\omega(\tau_0)|} = \frac{\delta\mathbf{l}(\tau)}{|\delta\mathbf{l}(\tau_0)|} \qquad (1.10)$$

(τ_0 is instant for the start of movement). Taking account of the expansion for component ω and the coordinates of individual particles from (1.10) an approximate relationship emerges

$$\frac{\omega_{1,3}^{(0)}(\tau)}{\omega_h^{(0)}(\tau_0)} = \frac{\delta l_{1,3}(\tau)}{\delta l_h(\tau_0)}, \qquad (1.11)$$

where $\omega_h^{(0)}(\tau_0) = \sqrt{\omega_1^{(0)2} + \omega_3^{(0)2}}\big|_{\tau=\tau_0}$; $\delta l_h(\tau_0) = \sqrt{(\delta l_1)^2 + (\delta l_3)^2}\big|_{\tau=\tau_0}$. By solving (1.9) for particles found on one vortex line it is possible to find the change in projection of its material elements on plane ξ, z, and by means of (1.11) to erect the distribution of horizontal vorticity components along this vortex line at any instant of time τ.

In the two-dimensional case ($\partial/\partial z = 0$) the boundary problem for a CL has a trivial solution coinciding with that known in two-dimensional CL theory [13, 14]: $V_2^{(0)} = V_c^{(0)}$, $V_1^{(0)} = U_c'Y$, $V_3^{(0)} = 0$ ($\omega_3^{(0)} = -U_c'$, $\omega_{1,2}^{(0)} = 0$). With this current line flow in the CL has a "cat's eye" shape. The nonlinear problem for one inclined wave is also reduced to the two-dimensional case (similar to [13]). With presence of a pair of inclined waves disturbances of vorticity $\omega_{1,3}$ are comparable with the vorticity for primary flow, i.e., U_c' and linear velocity in the reckoning system connected with waves do not coincide with $U_c'Y$. In addition, in the case of a nonlinear CL the first two equations of set (1.8) determine the product of divergence for horizontal velocity differing from zero: $d/dt\ (\partial V_1^{(0)}/\partial\xi + \partial V_3^{(0)}/\partial z) \neq 0$. In accordance with the continuity equation this leads to a change in velocity $V_2^{(0)}$ across the CL. An increase in deviation of $V_2^{(0)}$ from $V_c^{(0)}$ on approaching the resonance point is also detected in expansions (1.5).

In order to describe the reverse effect of vorticity disturbances in a CL on evolution of wave amplitude it is necessary to find the jump in coefficient A in (1.3), which leads to the problem of combining expansions for v_2 of the order $\varepsilon^{3/2}$. Movement of the main sequence with disturbances of vorticity in CL O(1) does not give jumps in a single one of the velocity components with transition through the CL, and consequently there are no jumps in Reynolds wave stresses governing the evolution of wave amplitudes. Therefore, the rate of change in amplitude will be of the same order with respect to ε, as in the case one two-dimensional or one inclined wave.

2. We consider evolution of vortex lines in a linear CL with presence in the stream of one of the wave triplets shown in Fig. 1. We shall assume that at initial instant $\tau_0 = 0$ disturbances of horizontal velocity in the resonance region $|Y| \lesssim 1$ are absent: $V_1^{(0)} = U_c'Y$, $V_3^{(0)} = 0$. In this way vortex lines governing the field of vorticity ω^0 have the form of straight lines drawn out across the flow, and in expression (1.1) we should place $|\omega^{(0)}(0)| = U_c'$ and $|\delta\mathbf{l}(0)| = |\delta\mathbf{l}_3(0)|$. Pressure distribution in the CL is presented as

$P^{(0)} = s_1 \cos(\alpha\xi - \beta z) + s_1 \cos(\alpha\xi + \beta z) + s_0 \cos(K\alpha\xi + \Delta)$, where $s_j = |2B_p(\alpha_j, \beta_j)|$ are wave amplitudes, Δ is phase shift (phases of inclined waves may always be excluded by changing the start of reading along axes ξ and z); K = 1 corresponds to a single-frequency triplet, K = 2 corresponds to a subharmonic triplet (see Fig. 1a, b). For solution of system (1.9) we introduce normalized variables.

$$\tau_N = (\Omega/2\pi)\tau,\ \xi_N = \alpha\xi/2\pi,\ z_N = \alpha z/2\pi, \quad u = U_c' d V_1^{(0)},\ w = U_c' d V_3^{(0)}, \tag{2.1}$$

where $\Omega = \alpha\sqrt{2s_1 + s_0}$ is a characteristic frequency for small oscillations of entrained particles; $d = \sqrt{2s_1 + s_0}/U_c'$ is characteristic size of the entrainment region with respect to Y, which is found with substitution of $V_2^{(0)}$ by its limiting value $V_c^{(0)}$. By substituting (2.1) in (1.9) and discarding index N for variables τ_n, ξ_n, z_N, we obtain

$$\frac{d\xi}{d\tau} = u,\ \frac{du}{d\tau} = -\frac{\partial P}{\partial \xi}, \quad dz/d\tau = w,\ dw/d\tau = -\partial P/\partial z, \tag{2.2}$$

$$P(\xi,\ z) = e \cos(2\pi\xi) \cos(2\pi\gamma z) + (1 - e)\cos(2\pi K\xi + \Delta).$$

Here $\gamma = \beta/\alpha$ is tangent of the slope for oblique waves; $e = 2s_1/(2s_1 + s_0)$.

System (2.2) has an energy integral $u^2/2 + w^2/2 + P(\xi, z) = E$. Each line for level of potential P = E is the boundary of the entrainment region in plane ξ, z for particles with energy E (movement is prohibited with P > E). We shall follow particles located with $\tau = 0$ in any vortex line of undisturbed flow: $u(0) = Y(0)/d = \text{const}$, $\xi(0) = \text{const}$, $w(0) = 0$ [z(0) is a parameter]. Any solution of (2.2) with these initial conditions is invariant with respect to transforms $\gamma z \to \gamma z + 1$ and $\xi \to \xi + 1$ or relative to double-substitutions $\gamma z \to n - \gamma z$, $w \to -w$ (n is a whole number), which relates to periodic repeatability of the trajectory with respect to ξ, z and their mirror symmetry relative to lines $\gamma z = n - 1/2$. Therefore, it is sufficient to plot the trajectory of particles found with $\tau = 0$ within the limits of one period with respect to ξ and z. The initial position of a vortex line prescribes nonuniform distribution of particle energy with respect to z. In this way in one vortex line there may be "entrained" and "flying" particles completing, respectively, finite and infinite movement in flow direction ξ.

In this vortex line there will be tension for material elements and rotation of them in the flow direction, which according to (1.11) leads to development of considerable linear vorticity. The condition of particle speedup in one vortex line may be obtained by analyzing the movement of particles with coordinates $\gamma z(0) = n$, $n - 1/2$, which move in plane ξ, z along straight lines $z = \text{const} = z(0)$ ($w \equiv 0$). Boundaries of the entrainment region for these particles in plane ($u^2(0)$, ξ) are determined by an expression in the form of $(1/2)u^2(0) = \max_\xi[P(\xi, z(0))] - P(\xi, z(0))$. In the case of a single-frequency triplet (K = 1) with $\Delta = 0$, $e < 0.5$ the reciprocal position of these boundaries is such that particle speedup in lines $\gamma z = n$, $n - 1/2$ with any $\xi(0)$ proceeds with departure to infinity of those particles which are in lines $\gamma z = n - 1/2$.

Movement of particles in one of these vortex lines is shown in Fig. 2 [$\gamma = 1$, $e = 0.3$, $u(0) = 1.5$, curves 1-4 relate to instants of time shifted by $\Delta\tau \approx 0.31$]. It can be seen that tension for material elements leads to formation of a Λ-shaped inflection on whose slopes considerable linear vorticity arises with opposite signs. With intersection by particles of planes of symmetry $\gamma z = n$, $n - 1/2$, in which $w = 0$, loops occur on vortex lines. The condition for impenetrability of these planes is not infringed, since there is a symmetrical original counterflow of particles from adjacent regions, which makes it possible to identify intersection of planes of symmetry with elastic reflection of particles falling on them.

Presented in Fig. 3 are ten vortex lines at instant $\tau = 0.65$, which with $\tau = 0$ were found in sections $\xi = 0.1(\ell - 1)$ ($\ell = 1$-10) in one level with respect to Y [$u(0) = 1.5$]. This picture shows that the concentration of linear vorticity occurs as a result of lines in which the speedup condition for liquid particles is fulfilled. Sections given in Figs. 2 and 3 for vortex lines should continue periodically, shifting the origin in the direction of axes ξ and γz with a period equal to unity. With $\Delta \neq 0$ or $e > 0.5$ in vortex lines with

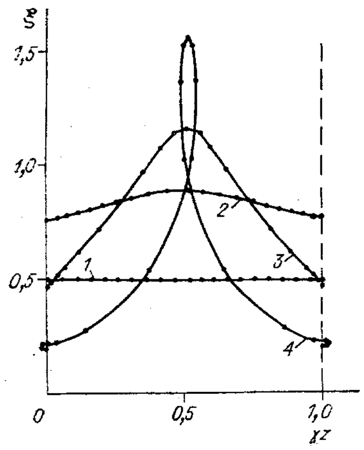

Fig. 2

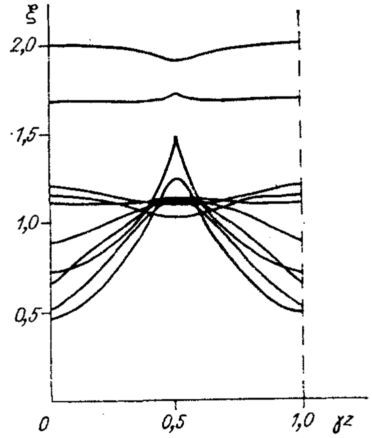

Fig. 3

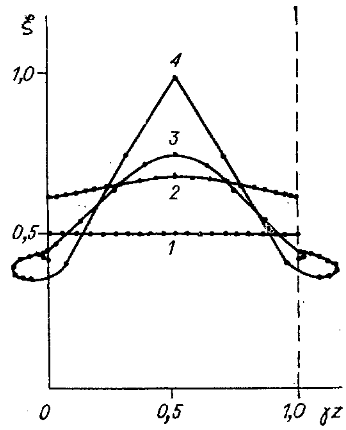

Fig. 4

different $\xi(0)$ the heads of inflections may also be found in straight lines $\gamma z = n$. To a considerable extent experimental data relate to the picture of lines with inflections of a single type occurring with $\Delta = 0$, $e < 0.5$ [4, 9]. Condition $\Delta = 0$ may be considered as the result of synchronizing two-dimensional and inclined waves with formation of a nonlinear vortex structure in the CL. Solutions of (2.2) with $\Delta = 0$ are invariant relative to substitutions $x \to -x$, $u \to -u$. Therefore, in vortex lines found with $\tau = 0$ beneath the layer of conformity [$u(0) < 0$] Λ-shaped inflections occur directed against the flow. Both types of inflection in vortex lines were also obtained in [20] with direct numerical modeling on ideal flow near a wall by means of a system of vortex filaments.

In the case of a subharmonic triplet (K = 2) the property of periodicity for location of trajectories with a subharmonic period is fulfilled by the property of symmetry of a checkered type [solutions of (2.2) are invariant to substitutions $\gamma z \to \gamma z + 1/2$, $\xi \to \xi + 1/2$]. Analysis of experimental data given in [18] gives a value $\Delta \approx \pi$. Movement of particles in a section of one vortex line with $\Delta = \pi$ is shown in Fig. 4 [$\gamma = 1$, $e = 0.5$, $u(0) = 1.1$; curves 1-4 correspond to instants shifted by $\Delta\tau = 0.22$]. The picture obtained should continue periodically, shifting the origin in the direction of axes ξ and γz with period 1. In addition, by considering the checkered symmetry in the position of the particle trajectory, this picture may be continued by shifting the origin along axes ξ and γz by one half simultaneously. Thus, for subharmonic interaction a checkered sequence of Λ-shaped inflections is typical.

The results obtained for a CL in ideal flow may be connected with the stationary (or quasistationary) CL established with reaction of waves in viscous flow. This relationship relates to the case of quite large wave amplitudes when the scale of a nonlinear CL exceeds the thickness of a linearly viscous CL [14]. Nonuniform tension of material elements for vortex lines leads to an increase in vorticity gradients across the CL and entry into play of viscous flow forces. It is natural to assume that in a CL a vortex structure occurs qualitatively similar to nonviscous flow in this stage of its development when viscous terms in CL equations are of a single order with nonviscous terms. These ideas are confirmed by the results of solving the nonviscous and steady viscous problems for a two-dimensional CL [14, 15].

From the analysis carried out it can be seen that concentration of vorticity in linear vortices is due to tension of material elements in vortex lines in a nonuniform pressure field across the flow occurring with resonance reaction of a two-dimensional wave with a pair of inclined waves. The theory explains certain regularities observed in experiments for visualizing flow in the transition region [6-9]. Linear vorticity has opposite signs in neighboring slopes of a Λ-vortex; in the case of a single-frequency wave triplet vortices are set up successively and it is possible for the head of the vortex to break away in the final stage of its development (see Fig. 2); with a subharmonic type of transition vortices build up in a checkered sequence and head separation is absent (see Fig. 4). At the same time, theory predicts formation of reverse vortices lying below the layer of conformity which are not observed in experiments. Comparison with results in [20] make it possible to conclude that development of reverse vortices is a result of using an ideal flow model. With wave amplitudes corresponding to formation of a strongly linear CL the region of entrained particles expands almost to the walls [15]. It may be assumed that absence of reverse vortices is connected with the viscous nature of flow close to the walls.

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EXPERIMENTAL MODEL OF A TORNADO

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UDC 532.527+551.515.3

In this work results are presented for an experimental study of movement of a fluid filling a cylindrical vessel moving with constant angular velocity, part of the surface of which oscillates in a prescribed way. It has been established that under certain conditions a system of vortices forms in the fluid. The main properties of these vortices are the oscillating nature of fluid movement in them, and the high level of vorticity markedly exceeding double the angular velocity of vessel rotation. In arranging experiments considerable use was made of data in [1, 2] where in a linear approximation information is given about actual oscillations of a solidly rotating cylindrical column of fluid. It was found that the properties of laboratory vortices are similar to those known for natural atmospheric vortices, i.e., tornadoes. The analogy established makes it possible to explain numerous facts caused by the occurrence of a tornado.

1. Experiments were carried out in a device whose diagram is given in Fig. 1. A transparent cylindrical vessel 1, in which the fluid was placed, rotated with constant angular velocity ω. Movement of the initially solidly rotating fluid was disturbed by means of generator 2, consisting of a disk, or a ring, or of a disk and a ring. The fluid surface 3 between disks, rings, and the side surface of the vessel is free. Disks and rings rotate together with the vessel, and in the vicinity of the free fluid surface they complete harmonic vertical oscillations with frequency $\omega_h = \omega$. By this method axisymmetrical inertial waves (zero harmonic for the amplitude coordinate) were created in the fluid. A resonance regime for wave excitation was used which made it possible to isolate the required mode and made it possible to obtain waves of considerable amplitude. In order to provide resonance the level of fluid in the vessel was chosen so that with a prescribed oscillation frequency for the generator the height of the fluid column equalled a whole number N of half-waves for the test mode. The edges of generator disks and rings moved over cylindrical surfaces, where vertical velocity v_z in the exciting wave, calculated by linear theory [1], returned to zero (see Fig. 1).

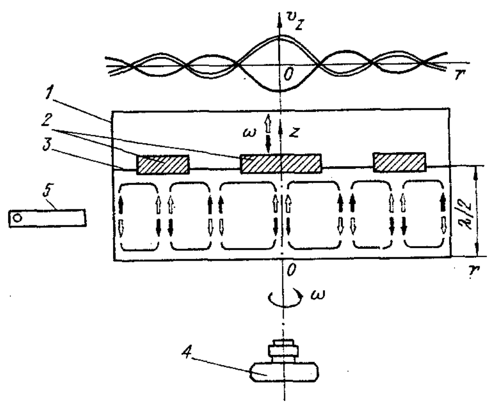

Fig. 1

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